

SELF-DUALITY AND MAXIMALLY HELICITY VIOLATING QCD AMPLITUDES

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I review some recent work that describes the close analogy between self-dual Yang–Mills amplitudes and QCD amplitudes with external gluons of positive helicity. This analogy is carried at tree level for amplitudes with two external quarks and up to one-loop for amplitudes involving only external gluons.

1. Introduction

Low-dimensional field theories are extensively investigated due to their greater tractability. Although the mathematical structure emerging from lower dimensional models is often worth some attention in itself, one really hopes to learn something about the physical world. This has often been the case: two-dimensional QED and QCD have taught us a lot on asymptotic freedom, three-dimensional Chern-Simons theory has played a crucial rôle in planar condensed matter systems.

Low-dimensional field theories are sometimes completely integrable. Moreover all known integrable models derive from the self-dual Yang–Mills (SDYM) equations. I will present in this article a physical application of these equations and show an explicit relation between a solution of the SDYM equations and some QCD amplitudes. The amplitudes that I will discuss involve external gluons with all the same helicity. Since these amplitudes are known to vanish at tree level, they are referred to as Maximally Helicity Violating (MHV) amplitudes. The notions of self-duality and positive helicity coincide for Maxwell (free) fields and are intimately connected in the case of non-Abelian gauge fields.¹ A link between MHV amplitudes and integrable models was pointed out by Nair.² Recently, Bardeen^{3,4} has shown the direct relevance of a SDYM solution in the calculation of MHV tree amplitudes. This analysis has then been extended to one-loop amplitudes.^{5,6}

The probe of physics beyond the Standard Model needs more and more accurate data, not only from experimentalists but also from theoreticians. The leading order of α_s is obtained by squaring tree-level scattering amplitudes but the next-to-leading order is also necessary. To this end people have developed ingenious methods to calculate tree and one-loop QCD amplitudes.⁷ After lengthy and complex calculations, some of these amplitudes turn out to be very simple. Tree amplitudes with all or all but one external gluons with positive helicity^a are zero,

$$A_n^{\text{tree}}(g_1^+, \dots, g_n^+) = 0, \quad (1)$$

$$A_n^{\text{tree}}(g_1^-, g_2^+, \dots, g_n^+) = 0. \quad (2)$$

^aBy convention, helicity is defined for outgoing particles.

Non trivial tree amplitudes^b are for example the Park-Taylor⁸ amplitudes with two negative helicity external gluons,

$$A_n^{\text{tree}}(g_1^-, g_2^-, g_3^+, \dots, g_n^+) = ig^4 \frac{\langle 12 \rangle^4}{\langle 12 \rangle \langle 23 \rangle \dots \langle n1 \rangle}. \quad (3)$$

Whereas amplitudes (1) vanish at tree level, their one-loop correction is non zero and has a very simple structure. The four-point function is:⁹

$$A_4^{\text{one-loop}}(g_1^+, g_2^+, g_3^+, g_4^+) = -N_c g^4 \frac{i}{48\pi^2} \frac{[12][34]}{\langle 12 \rangle \langle 34 \rangle}, \quad (4)$$

and their form is explicitly known for an arbitrary number of external gluons.¹⁰

In view of these results, several questions arise naturally. Their answers will be the object of this article:

1. Why is the form of these amplitudes so simple?
2. Is there an “effective model” for QCD restricted to positive helicity configurations which reproduces these amplitudes?
3. Is there a symmetry behind the vanishing of the tree amplitude (1) ?
4. Is the one-loop result (4) a manifestation of an anomaly?

At the present time, the few answers and the possible hints that we have are listed hereafter and will be developed in the following sections.

1. It is clear that helicity is the relevant factor. A positive helicity configuration is self-dual: a positive helicity electromagnetic wave has $\vec{E} = i\vec{B}$, which is nothing else than the self-dual equation $F_{\mu\nu} = i\epsilon_{\mu\nu\rho\sigma}F^{\rho\sigma}$.
2. Nair first showed² that the amplitude $A_n^{\text{tree}}(g_1^-, g_2^-, g_3^+, \dots, g_n^+)$ can be derived from the current algebra of a $k = 1$ WZNW model based on CP^1 . Keeping with the analogy between positive helicity and self-duality, Bardeen observed³ that some solutions of the SDYM equations reproduce the QCD amplitudes $A_n^{\text{tree}}(g_1^+, \dots, g_n^+)$. This observation can be extended^{5,6} to the one-loop amplitudes (4) using various quantizations of the SDYM equations. One of them⁶ is directly related to the self-dual sector of QCD, a truncation of $N = 4$ SYM theory in the light-cone formalism. It is interesting to notice that one of the other quantizations involves an action which is just a generalization¹¹ in four dimensions of the WZNW action, a possible link to the earlier work of Nair.
3. Up to now, the identification of the self-dual amplitudes with the MHV amplitudes has in no way simplified the actual calculations. One would like a deeper understanding of the vanishing of the tree amplitudes, for example using some symmetry argument. Indeed, the SDYM equations possess an infinite symmetry that forms an affine Lie algebra and it is suspected, although not proven, that this large symmetry is responsible for the vanishing of the tree amplitudes.

^bOnly the color-leading partial amplitudes are shown here. I also use the spinor notation where $\langle 12 \rangle = -(k_{1,0+z}/k_{1,x+iy} - k_{2,0+z}/k_{2,x+iy})k_{1,x+iy}k_{2,x+iy}/\sqrt{|k_{1,0+z}k_{2,0+z}|}$ and $[12] = \text{sign}(k_{1,0}k_{2,0})\langle 21 \rangle^*$. Details on color ordering and the spinor notation can be found elsewhere.^{7,5}

4. Assuming that the previous argument holds at tree level, one concludes that the one-loop results (4) should be a consequence of an anomaly.

These are the various pieces of the puzzle and we will now see how they fit together.

After this introduction, I review the SDYM equations, their solutions and their symmetries. I discuss the tree amplitudes in section 3 and the one-loop amplitudes in section 4. I conclude with some remarks in section 5.

2. SDYM equations

SDYM equations have real solutions only in Euclidean or (2+2)-signature spacetime. We here work in Minkowski spacetime, where they are:

$$F_{\mu\nu} = \frac{i}{2} \epsilon_{\mu\nu\rho\sigma} F^{\rho\sigma}, \quad (5)$$

and have complex solutions. It was remarked by Duff and Isham¹ that complex field configurations are naturally provided by off-diagonal matrix elements of the hermitian field operator. In the Yang-Mills case, they further showed that a solution to the equations of motion with definite duality is given by a matrix element between the vacuum and a coherent state with definite helicity. We shall come back to this point in the next section. In components,^c the SDYM equations read:

$$F_{0+z, x-iy} = 0, \quad F_{0-z, x+iy} = 0, \quad F_{0-z, 0+z} = F_{x+iy, x-iy}. \quad (6)$$

Dimensional reductions of these equations yield various (maybe even all¹²) integrable models. The (2 + 1)-dimensional Chern-Simons system coupled to self-dual matter and discussed elsewhere in these proceedings is such an example.

In Yang's approach,¹³ the two first equations are taken as two zero curvature conditions solved by:

$$\begin{aligned} -\frac{ig}{\sqrt{2}} A_{0+z} &= h^{-1} \partial_{0+z} h, & -\frac{ig}{\sqrt{2}} A_{x-iy} &= h^{-1} \partial_{x-iy} h, \\ -\frac{ig}{\sqrt{2}} A_{0-z} &= \bar{h}^{-1} \partial_{0-z} \bar{h}, & -\frac{ig}{\sqrt{2}} A_{x+iy} &= \bar{h}^{-1} \partial_{x+iy} \bar{h}. \end{aligned} \quad (7)$$

The last SDYM equation gives then:

$$\partial_{0-z} (H^{-1} \partial_{0+z} H) - \partial_{x+iy} (H^{-1} \partial_{x-iy} H) = 0, \quad H = h \bar{h}^{-1}. \quad (8)$$

This resembles a two-dimensional conserved current equation and is in fact obtained from an action similar to the WZNW action, first proposed by Donaldson¹⁴ and by Nair and Schiff,¹⁵

$$\begin{aligned} S_{\text{DNS}}(H) &= \frac{f_\pi^2}{2} \int d^4x \operatorname{tr} (\partial_{0+z} H \partial_{0-z} H^{-1} - \partial_{x-iy} H \partial_{x+iy} H^{-1}) \\ &+ \frac{f_\pi^2}{2} \int d^4x dt \operatorname{tr} ([H^{-1} \partial_{0+z} H, H^{-1} \partial_{0-z} H] - \\ &\quad [H^{-1} \partial_{x-iy} H, H^{-1} \partial_{x+iy} H]) H^{-1} \partial_t H. \end{aligned} \quad (9)$$

One checks that its β -function vanishes at one-loop and possibly at all order.¹⁶

^cI use the conventions $k_{0\pm z} = k_0 \pm k_z$ and $k_{x\pm iy} = k_x \pm i k_y$, with metric $g_{\mu\nu} = \operatorname{diag}(1, -1, -1, -1)$ and totally antisymmetric tensor $\epsilon_{0123} = 1$.

Another way to solve the SDYM equations is to go in the light-cone gauge $A_{0-z} = 0$, where the two last SDYM equations are solved by:

$$A_{x+iy} = 0, \quad A_{0+z} = \sqrt{2} \partial_{x+iy} \Phi, \quad A_{x-iy} = \sqrt{2} \partial_{0-z} \Phi, \quad (10)$$

in terms of a zero dimensional scalar field Φ . The remaining SDYM equation gives:

$$\partial^2 \Phi - ig [\partial_{x+iy} \Phi, \partial_{0-z} \Phi] = 0. \quad (11)$$

There are two actions associated to these equations of motion.

The first one was proposed by Leznov, Mukhtarov and Parkes.¹⁷ It does not introduce new field,

$$S_{\text{LPM}}(\phi) = f_\pi^2 \int d^4x \operatorname{tr} \left(\frac{1}{2} \partial \phi \cdot \partial \phi + \frac{ig}{3} \phi [\partial_{x+iy} \phi, \partial_{0-z} \phi] \right), \quad (12)$$

but is not real, explicitly breaks Lorenz invariance and has a non renormalizable interaction by power counting.

The second action, proposed by Chalmers and Siegel⁶, uses a Lagrange multiplier of dimension two to enforce Eq. (11),

$$S_{\text{SC}}(\phi, \Lambda) = - \int d^4x \operatorname{tr} \Lambda (\partial^2 \Phi - ig [\partial_{x+iy} \Phi, \partial_{0-z} \Phi]) \quad (13)$$

This action is also obtained after a truncation of $N = 4$ SYM theory in light-cone formalism^{18,19} (Λ and Φ correspond then to the highest and lowest components of the $N = 4$ chiral superfield). The auxiliary field Λ obeys the additional equation of motion:

$$\partial^2 \Lambda - ig [\partial_{x+iy} \Lambda, \partial_{0-z} \Phi] - ig [\partial_{x+iy} \Phi, \partial_{0-z} \Lambda] = 0, \quad (14)$$

and appears only in tree amplitudes. In fact, it is impossible to draw connected diagrams with more than one loop and thus this model only admits tree and one-loop corrections! However the Hamiltonian is not bounded by below so one should be careful in the full quantization of the model.

The symmetries $\Phi \rightarrow \Phi + \Lambda / f_\pi^2$ of the SDYM equations in their light-cone gauge form (11) are also described by Eq. (14). A one-parameter family of symmetries Λ_s is constructed from the following pair of recursion relations:²⁰

$$\begin{aligned} \partial_{0-z} \Lambda_{s+1} &= \partial_{x-iy} \Lambda_s - ig [\partial_{0-z} \Phi, \Lambda_s], \\ \partial_{x+iy} \Lambda_{s+1} &= \partial_{0+z} \Lambda_s - ig [\partial_{x+iy} \Phi, \Lambda_s]. \end{aligned} \quad (15)$$

These equations are compatible if Λ_s is a solution of (14). Moreover Λ_{s+1} is a symmetry if Φ is a solution of the SDYM equations. These symmetries are known to form an affine Lie algebra. Of course, the Λ 's are symmetries of the equations of motion and not necessarily of the action. Among all the Λ_s , only $\Lambda_0 = T^a$ and $\Lambda_1 = -ig [\phi, T^a]$ are true symmetries of the action. Nevertheless, the hierarchy (15) defines an infinite set of conserved currents whose classical expressions are:

$$\begin{aligned} \mathcal{J}_{s,0+z} &= \partial_{0+z} \Lambda_s - ig [\partial_{x+iy} \Phi, \Lambda_s], \quad \mathcal{J}_{s,x-iy} = \partial_{x-iy} \Lambda_s - ig [\partial_{0-z} \Phi, \Lambda_s], \\ \partial_\mu \mathcal{J}_s^\mu &= \frac{1}{2} ig [\Lambda_{s-1}, \partial^2 \Phi - ig [\partial_{x+iy} \Phi, \partial_{0-z} \Phi]] = 0. \end{aligned} \quad (16)$$

Since these currents are only known for classical solutions Φ , one can only derive tree level identities and not true Ward identities relating Green functions of different orders in \hbar .

3. Tree amplitudes

We now compare the tree amplitudes of these SDYM models with the QCD ones. Since the three SDYM actions have equivalent equations of motion they have equivalent tree amplitudes. Namely, there is a direct relation between tree amplitudes and classical solutions. An amplitude is the on-shell truncation of a connected Green function. A connected tree Green function is generated by the Legendre transform $W(J)$ of the classical action $S(\phi)$,

$$\left. \frac{\delta S(\phi)}{\delta \phi(x)} \right|_{\phi=\Phi_J} + J(x) = 0, \quad (17)$$

$$W(J) = S(\Phi_J) + \int dx \Phi_J(x) J(x), \quad (18)$$

$$\langle \phi(x_1) \cdots \phi(x_{n+1}) \rangle_c = \left. \frac{i \delta^{n+1} W(J)}{i \delta J(x_1) \cdots i \delta J(x_{n+1})} \right|_{J=0}. \quad (19)$$

Since the classical solution Φ_J in presence of a source J is given by the first variation of the generating functional $W(J)$, we have:

$$\langle \phi(x_1) \cdots \phi(x_{n+1}) \rangle_c = \left. \frac{\delta^n \Phi_J(x_{n+1})}{i \delta J(x_1) \cdots i \delta J(x_n)} \right|_{J=0}. \quad (20)$$

The classical solution Φ_J is an infinite series in J whose coefficients are the connected Green functions. In our case, the equations of motion with source read:

$$\partial^2 \Phi_J - ig [\partial_{x+iy} \Phi_J, \partial_{0-z} \Phi_J] + J = 0 \quad (21)$$

whose solution is obtained by inverting the differential operator ∂^2 with the Feynman propagator $-1/(k^2 + i\epsilon)$.

Following Duff and Isham,¹ let us consider a coherent in-state (the minus sign is introduced to match earlier conventions⁵) $|\tilde{j}\rangle = \sum_{m=0}^{\infty} \frac{1}{m!} |(-j)^m\rangle$ based on a positive energy on-shell wave,

$$j(x) = -i \sum_{j=1}^n T^{a_j} e^{-ik_j x} f(k_j), \quad (22)$$

(the function $f(k)$ has support on the light-cone). By LSZ reduction, we construct the connected tree matrix element of the interacting field operator $\hat{\phi}$ between this state and the vacuum,

$$\begin{aligned} \langle 0 | \hat{\phi}(k) | \tilde{j} \rangle_c^{\text{tree}} &= \sum_{m=1}^{\infty} \frac{i^m}{m!} \int \frac{d^4 p_1}{(2\pi)^4} \cdots \frac{d^4 p_m}{(2\pi)^4} \\ &\times p_1^2 j(-p_1) \cdots p_m^2 j(-p_m) \langle \phi(p_1) \cdots \phi(p_m) \phi(k) \rangle_c^{\text{tree}}. \end{aligned} \quad (23)$$

We are interested in graphs with a given number of external leg, say $n+1$:

$$\left\langle \phi(k) \right\rangle_{1 \dots n} \equiv (-ik_1^2) f(k_1) \cdots (-ik_n^2) f(k_n) \left\langle \phi(k_1) \cdots \phi(k_n) \phi(k) \right\rangle \Big|_{k_1^2 = \dots = k_n^2 = 0}. \quad (24)$$

They are readily obtained from the matrix element (23). One inserts (22) and keeps the terms proportional to $g^{n-1} f(k_1) \cdots f(k_n)$. A tree amplitude with $(n+1)$ external legs is then simply $(-ik^2) \left\langle \phi(k) \right\rangle_{1 \dots n} \Big|_{k^2=0}$.

The matrix element (23) is connected to the solution Φ_J through the following identity:¹

$$\langle 0 | \hat{\phi}(k) | \tilde{j} \rangle_c^{\text{tree}} = \exp \left[\int \frac{d^4 p}{(2\pi)^4} p^2 j(p) \frac{\delta}{\delta J(-p)} \right] \Phi_J(k) \Big|_{J=0} = \Phi_{J(p)=p^2 j(p)}(k). \quad (25)$$

We need thus to find a solution to the classical equations of motion with an on-shell source $J(p) = p^2 j(p)$. The solution is written as a series in the coupling constant g ,

$$\Phi_J(x) = \sum_{m=1}^{\infty} \Phi_J^{(m)}(x), \quad \Phi_J^{(m)} \propto g^{m-1}. \quad (26)$$

Using the explicit form of $j(p)$, we prove by recurrence:^{3,5}

$$\begin{aligned} \Phi^{(n)}(x) &= -ig^{n-1} \sum_{\substack{\text{permutations} \\ \text{of } (1 \dots n)}} T^{a_1} \dots T^{a_n} e^{-i(k_1 + \dots + k_n)x} f(k_1) \dots f(k_n) \\ &\times (Q_1 - Q_2)^{-1} (Q_2 - Q_3)^{-1} \dots (Q_{n-1} - Q_n)^{-1} + \dots \end{aligned} \quad (27)$$

with $Q_i = k_{i,x+iy}/k_{i,0+z}$. The piece written here involves all momenta k_j ($j = 1, \dots, n$) and, as mentioned before, is equal to the current amplitude (24). The additional terms denoted by the triple dots (as well as the other $\Phi^{(m)}$, $m \neq n$) correspond to tree diagrams having two or more legs with the same momentum.^{5,21}

The remarkable property of this solution is that it does not involve the multi-particle poles appearing in the intermediate states. In particular there is no pole in $k^2 = (k_1 + \dots + k_n)^2$ so that the tree amplitudes (obtained by multiplying (24) by k^2 and taking the limit $k^2 \rightarrow 0$) vanish.

The MHV tree amplitudes (1) vanish in exactly the same way. The Berends and Giele relations²² derived in QCD coincide for MHV configurations with the recurrence relations obtained here. For $f(k) = Q/k_{x+iy}$, we get the following identity⁵ between the tree SDYM current (24) and the corresponding tree MHV current in the light-cone gauge as calculated for example by Mahlon *et al.*:²³

$$\begin{aligned} \left\langle A_{0+z}(k) \right\rangle_{1+\dots n+}^{\text{tree}} &= -i\sqrt{2} k_{x+iy} \left\langle \phi(k) \right\rangle_{1\dots n}, & \left\langle A_{x+iy}(k) \right\rangle_{1+\dots n+}^{\text{tree}} &= 0, \\ \left\langle A_{x-iy}(k) \right\rangle_{1+\dots n+}^{\text{tree}} &= -i\sqrt{2} k_{0-z} \left\langle \phi(k) \right\rangle_{1\dots n}, & \left\langle A_{0-z}(k) \right\rangle_{1+\dots n+}^{\text{tree}} &= 0. \end{aligned} \quad (28)$$

[compare with the self-dual *Ansatz* (10)].

It is worth mentioning that the discussion extends to tree amplitudes with two quark lines.³ A right-handed quark current $\Psi^\alpha(1, \dots, n; p)$ [resp. left-handed antiquark current $\bar{\Psi}^\alpha(q; 1, \dots, n)$] consists of n on-shell positive helicity gluons, a right-handed quark with on-shell momentum p [resp. a left-handed antiquark with on-shell momentum q] and an off-shell antiquark [resp. an off-shell quark]. These currents verify Berends-Giele type recurrence relations²³ which are essentially a Dirac equation with a gauge field (28). Consider then:

$$\begin{aligned} \not{D}\Psi - \frac{ig}{\sqrt{2}} \not{A}_{\text{SD}}\Psi &= 0, \\ \bar{\Psi} \overleftarrow{\not{D}} - \frac{ig}{\sqrt{2}} \bar{\Psi} \not{A}_{\text{SD}} &= 0. \end{aligned} \quad (29)$$

With the representation (10) for the self-dual gauge field, the right-handed quark current equation forces the Weyl spinor Ψ to have the form:

$$\Psi = i \begin{pmatrix} \partial_{0-z} \\ -\partial_{x+iy} \end{pmatrix} \Xi, \quad (30)$$

where the scalar field Ξ obeys the second order differential equation:

$$\partial^2 \Xi - ig \partial_{x+iy} \Phi \partial_{0-z} \Xi + ig \partial_{0-z} \Phi \partial_{x+iy} \Xi = 0. \quad (31)$$

Like Φ , the field Ξ is expanded in powers of g . For the self-dual solution $\Phi_{J(p)=p^2 j(p)}$, one finds:

$$\begin{aligned} \Xi &= \sum_{m=1}^{\infty} \Xi^{(m)}, \quad \Xi^{(m)} \propto g^{m-1}, \\ \Xi^{(m)}(k) &= (ig)^{m-1} \int \frac{d^4 p_1}{(2\pi)^4} \cdots \frac{d^4 p_m}{(2\pi)^4} (2\pi)^4 \delta(p_1 + \cdots + p_m - k) \\ &\quad \times j(p_1) \cdots j(p_{m-1}) \Xi^{(0)}(p_m) (Q_1 - Q_2)^{-1} (Q_2 - Q_3)^{-1} \cdots (Q_{m-1} - Q_m)^{-1}, \\ p^2 \Xi^{(0)}(p) &= 0. \end{aligned} \quad (32)$$

With the plane wave *Ansatz* (22) for $j(p)$ and the choice $\Xi^{(0)}(p) = \sqrt{|p_{0+z}|}/p_{x+iy}$, one recovers the MHV quark current $\Psi^{\dot{\alpha}}(1, \dots, n; p)$ of Mahlon *et al.*²³

Similarly each of the two components $\bar{\Psi}^{\alpha}$ of the left-handed antiquark current verifies the same second order differential equation:

$$\partial^2 \bar{\Psi}^{\alpha} - ig \partial_{x+iy} \Phi \partial_{0-z} \bar{\Psi}^{\alpha} + ig \partial_{0-z} \Phi \partial_{x+iy} \bar{\Psi}^{\alpha} = 0, \quad (33)$$

with solution:

$$\begin{aligned} \bar{\Psi} &= \sum_{m=1}^{\infty} \bar{\Psi}^{(m)}, \quad \bar{\Psi}^{(m)} \propto g^{m-1}, \\ \bar{\Psi}^{(m)}(k) &= (ig)^{m-1} \int \frac{d^4 p_1}{(2\pi)^4} \cdots \frac{d^4 p_m}{(2\pi)^4} (2\pi)^4 \delta(p_1 + \cdots + p_m - k) \\ &\quad \times \bar{\Psi}^{(0)}(p_1) j(p_2) \cdots j(p_m) (Q_1 - Q_2)^{-1} (Q_2 - Q_3)^{-1} \cdots (Q_{m-1} - Q_m)^{-1}, \\ \bar{\Psi}^{(0)}(q) &= \begin{pmatrix} -q_{x+iy} \\ q_{0+z} \end{pmatrix} g(q), \quad q^2 g(q) = 0. \end{aligned} \quad (34)$$

For $g(q) = 1/\sqrt{|q_{0+z}|}$, the QCD antiquark current²³ $\bar{\Psi}^{\alpha}(q; 1, \dots, n)$ is also recovered.

It is straightforward to find an action of two scalar fields $\bar{\psi}, \xi$ that admits Eqs. (31) and (33) as equations of motion:

$$S_{\text{matter}}(\Phi, \bar{\psi}, \xi) = \int d^4 x \frac{1}{2} \bar{\psi} (\partial^2 \xi - ig \partial_{x+iy} \Phi \partial_{0-z} \xi + ig \partial_{0-z} \Phi \partial_{x+iy} \xi), \quad (35)$$

where Φ is the self-dual field (11). If Φ is to be dynamical, one can consider $S_{SC}(\phi) + S_{\text{matter}}(\phi, \bar{\psi}, \xi)$. Notice that the coupling to the two other actions would

modify the equation of motion for ϕ . It is also possible to extend the (classical) SDYM symmetries (14). Eq. (31) is invariant under $\Xi \rightarrow \Xi + \Delta$ if the shift Δ satisfies:

$$\partial^2 \Delta - ig \partial_{x+iy} \Phi \partial_{0-z} \Delta + ig \partial_{0-z} \Phi \partial_{x+iy} \Delta - \frac{ig}{f_\pi^2} \partial_{x+iy} \Lambda \partial_{0-z} \Xi + \frac{ig}{f_\pi^2} \partial_{0-z} \Lambda \partial_{x+iy} \Xi = 0. \quad (36)$$

This equation is solved by a one-parameter family of symmetry generators Δ_s ,

$$\begin{aligned} \partial_{0-z} \Delta_{s+1} &= \partial_{x-iy} \Delta_s - ig \partial_{0-z} \Phi \Delta_s + \frac{ig}{f_\pi^2} \Lambda_s \partial_{0-z} \Xi, \\ \partial_{x+iy} \Delta_{s+1} &= \partial_{0+z} \Delta_s - ig \partial_{x+iy} \Phi \Delta_s + \frac{ig}{f_\pi^2} \Lambda_s \partial_{x+iy} \Xi. \end{aligned} \quad (37)$$

which complete the previous Λ_s given in (15).

4. One-Loop Amplitudes

To go beyond tree level, one needs to consider the full actions. I will only discuss the gauge sector and thus do not include the matter action introduced at the end of the last section. Since the three SDYM actions are different, their S -matrix are different too. We will see in this section that their one-loop amplitudes are nevertheless equal and moreover coincide with one-loop MHV amplitudes.

I use the following construction of a one-loop amplitude. Take a truncated tree Green function with all legs on-shell except two which are then sewn together with a propagator. Each insertion on the loop corresponds to the multiplication by a factor:

$$- \frac{1}{f_\pi^2} \frac{\delta^2 S_{\text{DNS}}(H)}{H^{-1} \delta H(k_1) H^{-1} \delta H(k_2)} \Big|_{\text{classical solution}}, \quad (38)$$

$$\frac{1}{f_\pi^2} \frac{\delta^2 S_{\text{LMP}}(\phi)}{\delta \phi(k_1) \delta \phi(k_2)} \Big|_{\text{classical solution}}, \quad (39)$$

$$\left(\begin{array}{cc} \frac{\delta^2 S_{\text{SC}}(\phi, \Lambda)}{\delta \phi(k_1) \delta \phi(k_2)} & \frac{\delta^2 S_{\text{SC}}(\phi, \Lambda)}{\delta \phi(k_1) \delta \Lambda(k_2)} \\ \frac{\delta^2 S_{\text{SC}}(\phi, \Lambda)}{\delta \Lambda(k_1) \delta \phi(k_2)} & 0 \end{array} \right) \Big|_{\text{classical solutions}}. \quad (40)$$

This amounts to compare the one-loop effective actions of the three models.⁶ Taking into account that:

$$\begin{aligned} H^{-1} \partial_{0+z} H &= -ig \partial_{x+iy} \Phi, \\ H^{-1} \partial_{x-iy} H &= -ig \partial_{0-z} \Phi, \end{aligned} \quad (41)$$

one verifies that (38), (39) and (40) indeed match, up to a factor of two for S_{SC} (it has twice as many fields than the other actions), and thus establishes the equivalence of the three SDYM models at one-loop order.

Before comparing these results to QCD, I would like to indicate the analogy between the generators of the (classical) SDYM symmetries and the tree currents with two off-shell legs that we just used. The latter are generated by the two-point off-diagonal matrix element:

$$\begin{aligned} \langle 0 | \hat{\phi}(k) \hat{\phi}(q) | \tilde{j} \rangle_c^{\text{tree}} &= \sum_{m=1}^{\infty} \frac{i^m}{m!} \int \frac{d^4 p_1}{(2\pi)^4} \cdots \frac{d^4 p_m}{(2\pi)^4} \\ &\quad \times p_1^2 j(-p_1) \cdots p_m^2 j(-p_m) \langle \phi(p_1) \cdots \phi(p_m) \phi(k) \phi(q) \rangle_c^{\text{tree}}. \\ &= \frac{\delta \Phi_J(k)}{i \delta J(-q)} \Big|_{J(p)=p^2 j(p)}. \end{aligned} \quad (42)$$

One observes that $\delta\Phi_J(k)/i\delta J(-q)$ obeys the same equation (14) than the symmetry generators Λ . Notice moreover that the solution to these equations coincides⁵ with the generalized QCD current defined by Mahlon *et al.*^{23,5}

What about the MHV one-loop amplitudes as derived from QCD? In principle one should compute the one-loop effective action for QCD. However the comparison with the SDYM effective actions is difficult owing to the tensor structure that appears in the loop. It is much easier to use the one-loop equivalence between MHV QCD and scalar QCD.^{24,5} The MHV one-loop amplitudes (4) can be derived from the action:

$$S_{\text{QCD}} = \int d^4x \left(-\frac{1}{4} \text{tr} F_{\mu\nu} F^{\mu\nu} + \frac{1}{2} \text{tr} D_\mu \chi D^\mu \chi \right), \quad (43)$$

by keeping the gauge field classical and letting the field χ run in the loop. A simple verification, using the tree result (28), shows that the factor:

$$\left. \frac{\delta^2 S_{\text{QCD}}}{\delta\chi(k_1) \delta\chi(k_2)} \right|_{\text{classical solutions}} \quad (44)$$

coincides with (38), (39) and (40).

5. Conclusions

In this article I have reviewed how one-loop MHV amplitudes arise from a SDYM system. In fact three SDYM models have been proposed. They coincide up to one-loop but differ at higher order. It is believed that none of them will reproduce higher order QCD amplitudes. Nevertheless, one still thinks that they might provide a quicker way to derive one-loop results. This hope is based on the observation that the symmetries of the SDYM equations play a role in the derivation of one-loop SDYM amplitudes. According to Bardeen suggestion,³ the form of one-loop amplitudes might be a consequence of an anomaly in these symmetries. In the verification of this conjecture, one needs to better understand how to lift the classical SDYM symmetries to the quantum theory. Moreover it is not clear whether the existence of an anomaly is compatible with a zero beta function in the DNS model. This latter model has also been proposed to describe the low energy matter sector of the $N = 2$ heterotic string.²⁵ The $N = 2$ heterotic string has been shown to have vanishing amplitudes at all orders except the three-point amplitude. The apparent inconsistency with the non-vanishing one-loop amplitudes described here leads me to the following two remarks. First, the spacetime signature is here $(3+1)$ whereas in the $N = 2$ string it is $(2+2)$. Second, the low energy action of the $N = 2$ heterotic string also possesses a self-dual gravity sector^d which couples non trivially to the matter sector. It is not impossible that the sum of the matter contribution, the gravity contribution as well as the contribution coming from the mixing of the two add up to zero. Coming back to QCD, we have seen that it is possible to include two quarks lines in the amplitudes at the classical level and probably also at one-loop order.

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^dSelf-dual gravity can presumably be connected in the some way to MHV gravity amplitudes.

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